5.9 Short-rate appendix

Of all the formulas in the book, the expression for $g(x, t, T)$ at the top of page 152 has undoubtedly caused confusion for the greatest number of readers.

The reason for this is not that the formula is fundamentally hard, but rather it is derived from the preceding text only by a rather long argument which was omitted for reasons of readability.

So many people have asked for an explanation and description of the missing steps, that it seems a good idea to lay them out here.

There are four separate integrations to perform.

One: Finding the short rate

The Ho and Lee model box on page 151 contains the SDE for the short-rate. We can integrate this equation between times t and s to get the value of the short rate at time s, given its value at time t,

$$r_s = r_t + \sigma \int_t^s dW_u + \int_t^s \theta_u \, du.$$

The Brownian term can be simplified, as the integral of the differential of Brownian motion is just Brownian motion itself, so that

$$r_s = r_t + \sigma W_{s-t} + \int_t^s \theta_u \, du.$$

We notice now that r_s is a normal random variable. (Here we have also identified the Brownian motion W_{s-t} with its increment $W_s - W_t$. See page 48.)

Two: Integrating the short rate

Now we can integrate the short rate between times t and T which gives us

$$\int_t^T r_s \, ds = (T-t)r_t + \sigma \int_t^T W_{s-t} \, ds + \int_t^T (T-s)\theta_s \, ds.$$

In other words, the integral of the short-rate is a normal random variable, conditional on r_t, with mean μ and variance v^2. The mean μ is equal to

$$\mu = (T-t)r_t + \int_t^T (T-s)\theta_s \, ds.$$

Three: Calculating the variance

The variance v^2 is equal to

$$v^2 = \mathbb{E}_{\mathbb{Q}}\left(\left(\sigma\int_0^{T-t} W_u\,du\right)^2\right).$$

This can be evaluated as

$$v^2 = 2\sigma^2\int_0^{T-t}\int_0^s \mathbb{E}_{\mathbb{Q}}(W_s W_u)\,du\,ds,$$

which equals

$$v^2 = 2\sigma^2\int_0^{T-t}\int_0^s u\,du\,ds = \tfrac{1}{3}\sigma^2(T-t)^3.$$

Above, we used the fact that $\mathbb{E}(W_s W_u) = \min(s,u)$, which follows from the fact the Brownian increments are independent of their starting-value.

Four: Evaluating g

Now we are ready to evaluate the expression for g which is given on page 150,

$$g(x,t,T) = -\log\mathbb{E}_{\mathbb{Q}}\left(\exp(-N(\mu,v^2))\right).$$

This is equal to

$$g(x,t,T) = -\log\left(\exp(-\mu+\tfrac{1}{2}v^2)\right) = \mu - \tfrac{1}{2}v^2.$$

This follows from the useful result that $\mathbb{E}\big(\exp(N(\mu,v^2)\big) = \exp(\mu+\frac{1}{2}v^2)$. When we substitute in our expressions for μ and v^2 above, we finally achieve the desired formula:

$$g(x,t,T) = x(T-t) + \int_t^T (T-s)\theta_s\,ds - \tfrac{1}{6}\sigma^2(T-t)^3.$$

And so we are finished.