4.6 Exotic options

Here the fun begins. Standard options such as calls and forwards are well established. They have liquid markets, and may be traded on an exchange. Their prices are fairly well determined and margins are competitive. This gives an incentive to develop more complex instruments either to extend a bank's product range or to meet the hedging and speculative needs of its clients.

Such intricate contracts are labelled exotics, and are often grown and priced in the hothouse of quantitative analysis. But the principles of pricing and hedging exotics are exactly the same as those for basic options. The actual calculations may be more involved, but the methodology remains the same. We begin with a class of exotics which are as computationally easy as the simplest call option.

Terminal-value exotics

If the value of the exotic, maturing at time T, depends only on the price of the underlying security at that time, then life is straightforward. The price of the option at time t, paying off $f(S_T)$ at maturity T, is

$$V_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(f(S_T) \mid \mathcal{F}_t\right),$$

where interest rates are constant at r and $\mathbb{Q}$ is the measure under which $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale. Under the Black-Scholes model, or any other Markovian price process model, the price V_t will only depend on the current price of the stock.

Lookbacks and barriers

Both these families of exotics depend on the minimum or maximum of the underlying stock price. Let the minimum process S_* be

$$S_*(t) = \min\{S_u : 0 \leqslant u \leqslant t\},$$

and the maximum S^* be

$$S^*(t) = \max\{S_u : 0 \leqslant u \leqslant t\}.$$

These are both continuous monotone random processes (S_* is decreasing, S^* increasing), both starting at S_0.

A simple lookback call gives the right to buy a unit of stock at time T for a price equal to the minimum achieved by the stock up to time T. That is, the payoff is

$$X = S_T - S_*(T).$$

Barrier options give some right, such as a call option, which can only be exercised if the stock has crossed a preset barrier level at some time before maturity. Other flavours are only exercised if the stock does not cross the barrier. A 'down-and-in' call on a stock pays off $(S_T - k)^+$ only if the stock crossed the level c (where c is below S_0) some time before time T. The payoff is

$$X = I\big(S_*(T) \leqslant c\big)(S_T - k)^+.$$

This expression works because, given S_t is a continuous process, the stock has crossed the line c at some time before T if and only if the minimum value of the stock up to time T is less than or equal to c. So we can rewrite the barrier condition as the indicator function $I\big(S_*(T) \leqslant c\big)$.

Other varieties include the lookback put (with payoff $S^*(T) - S_T$), the 'down-and-out' barrier (payoff factor $I(S_*(T) \geqslant c)$), and similar 'up-and-in' and 'up-and-out' barriers. All of these payoffs are functions only of the terminal-value price S_T and one of the minimum and maximum prices $S_*(T)$ and $S^*(T)$. The price of such an option X, as usual, is given abstractly by the formula

$$\mathbb{E}_{\mathbb{Q}}\left(e^{-rT}X\right).$$

To use this formula, all we need is the joint distribution, under the martingale measure $\mathbb{Q}$, of $(S_T, S_*(T))$ and of $(S_T, S^*(T))$.

The Reflection Principle

Suppose we have a Black-Scholes world where W_t is $\mathbb{P}$-Brownian motion and

$$S_t = S_0 \exp\big(\sigma W_t + \mu t\big).$$

Suppose that μ happens to be exactly zero. Then in this case

$$S_t = S_0 \exp(\sigma W_t),$$

and the minimum of the stock S, $S_*(t)$ is the same as the exponential of the minimum of W, in that

$$S_*(t) = S_0 \exp(\sigma W_*(t)).$$

Because the function $\exp(\cdot)$ is one-to-one and increasing, the process S_t can hit a particular level c if and only if the process W_t hits a corresponding level (equal to $\log(c/S_0)$). Similarly S_t achieves a minimum value of c if and only if W_t achieves a corresponding minimum value.

The problem of finding the distribution of $(S_T, S_*(T))$ has been transformed into finding that of $(W_T, W_*(T))$, a problem which has been solved in probability theory.

The Reflection Principle

$$\mathbb{P}(W_T \in dy;\ W_*(T) \leqslant b) = p_T(0, 2b - y)\, dy, \qquad y > b,$$

where b is negative and $p_T(x, y)$ is the Brownian transition density

$$p_T(x, y) = \frac{1}{\sqrt{2\pi T}} \exp\bigl(-\tfrac{1}{2T}(y - x)^2\bigr).$$

This is true because the probability that W goes from 0 to y and crosses the line at height b is the same as the probability that W goes from 0 to $2b - y$ (the image of y reflected in the line b).

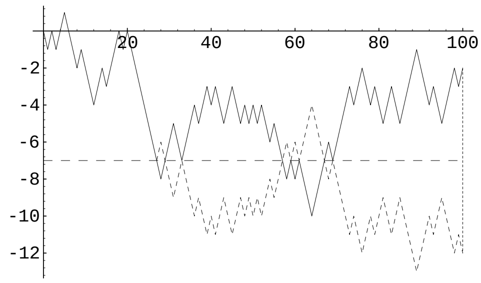

Figure 4.1: A random walk and its reflection in the line at height -7

There is a complete correspondence between paths from 0 to y which cross b, and paths from 0 to $2b - y$. Reflect the path in the line b at all times after it first hits.

In full

$$\mathbb{P}(W_t \in dy;\ W_*(T) \leqslant b) = \begin{cases} p_T(0,y)\,dy & \text{if } y \leqslant b, \\ p_T(2b,y)\,dy & \text{if } y > b. \end{cases}$$

This, of course, only works under our original condition that $\mu = 0$, which will not hold in general for either the original real world measure or the martingale measure. We can, however, use our C-M-G technology not only to switch to the martingale measure, but also to switch temporarily to the $\mu = 0$ measure.

Finding the martingale measure probabilities

If $\mathbb{Q}$ is the martingale measure, then

$$S_t = S_0 \exp\bigl(\sigma \tilde{W}_t + (r - \tfrac{1}{2}\sigma^2)t\bigr),$$

where $\tilde{W}$ really is $\mathbb{Q}$-Brownian motion. Then there is an equivalent measure $\mathbb{P}$ such that

$$W_t = \tilde{W}_t + \sigma^{-1}(r - \tfrac{1}{2}\sigma^2)t$$

is $\mathbb{P}$-Brownian motion, so that under $\mathbb{P}$

$$S_t = S_0 \exp(\sigma W_t).$$

The change of measure factor is

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(a\tilde{W}_T + \tfrac{1}{2}a^2T) = \exp(aW_T - \tfrac{1}{2}a^2T),$$

where a is $\sigma^{-1}(r - \frac{1}{2}\sigma^2)$. Very conveniently this change of measure only depends on the terminal value of the process W_T.

By property (i) of the Radon-Nikodym derivative in chapter three, the martingale measure $\mathbb{Q}$-probability of the event $\{W_T \in dy;\ W_*(T) \leqslant b\}$, will be the $\mathbb{P}$ probability of that event multiplied by $d\mathbb{Q}/d\mathbb{P}$ evaluated at $W_T = y$. Using the algebraic identity that

$$e^{ay - \frac{1}{2}a^2T} p_T(x,y) = e^{ax} p_T(x + aT, y), \qquad \text{for all } x,$$

we find that

$$\mathbb{Q}(W_T \in dy;\ W_*(T) \leqslant b) = \begin{cases} p_T(aT,y)\,dy & \text{if } y \leqslant b, \\ e^{2ab} p_T(2b + aT, y)\,dy & \text{if } y > b. \end{cases}$$

Or in terms of densities (for b negative),

$$\mathbb{Q}(W_T \in dy;\ W_*(T) \in db) = \frac{2e^{2ab}}{T}|y-2b|\, p_T(2b+aT, y)\, dy\, db, \quad \text{for } y > b.$$

Exactly the same formula holds for W^*, except that b is positive and y must be less than b. We can now calculate the price of any option which depends just on the terminal stock value and either the minimum or the maximum. For instance, the price now of an option which pays off $X = g(S_T, S_*(T))$ at time T is

$$\begin{aligned} V_0 &= \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}X\right) \\ &= e^{-rT}\int_{b=-\infty}^{0}\int_{y=b}^{\infty} g\left(S_0 e^{\sigma y}, S_0 e^{\sigma b}\right) \mathbb{Q}(W_T \in dy;\ W_*(T) \in db). \end{aligned}$$

Example — Down-and-In Call Option

Under the Black-Scholes model of $X_t = X_0 \exp(\sigma \bar{W}_t + \mu t)$, where $\bar{W}_t$ is Brownian motion under the real world measure, there is a new measure $\mathbb{P}$ with Brownian motion $W_t = \bar{W}_t + (\mu/\sigma)t$, under which $S_t = S_0 \exp(\sigma W_t)$.

We can now price a down-and-in call with payoff

$$X = (S_T - k)^+ I\big(S_*(T) \leqslant c\big),$$

where c is less than k. We can rewrite X, using $S_t = S_0 \exp(\sigma W_t)$, as

$$X = \left(S_0 e^{\sigma W_T} - k\right)^+ I\left(W_*(T) \leqslant \tfrac{1}{\sigma}\log\tfrac{c}{S_0}\right).$$

Writing a for $\sigma^{-1}(r - \frac{1}{2}\sigma^2)$, b for $\sigma^{-1}\log\frac{c}{S_0}$, and y_0 for $\sigma^{-1}\log\frac{k}{S_0}$, then

$$V_0 = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}X\right) = e^{-rT}\int_{y_0}^{\infty}\left(S_0 e^{\sigma y} - k\right)\mathbb{Q}\left(W_t \in dy;\ W_*(T) \leqslant b\right).$$

Using the expression above for the $\mathbb{Q}$-probability, this integral can be evaluated as

$$V_0 = e^{-rT}\left(\frac{c}{S_0}\right)^{2r/\sigma^2 - 1}\left\{F\Phi\left(\frac{\log\frac{F}{k} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F}{k} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\},$$

where F is given the value $e^{rT}c^2/S_0$. This is just the Black-Scholes formula along with a multiplying correction factor and the forward price replaced by F as given by the formula.

Example — Lookback Call

Under the same model as above, this option pays off

$$X = S_T - S_*(T)$$

at maturity T. Its value now is

$$V_0 = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}X\right) = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}S_T\right) - \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}S_*(T)\right).$$

The first of the two terms on the right is just S_0, as $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale. To calculate the second term $S_*(T) = S_0\exp(\sigma W_*(T))$, all we need is the distribution of $W_*(T)$, which is given by integrating the appropriate line above over y. We get that

$$\mathbb{Q}(W_*(T) \leqslant b) = \Phi\left(\frac{b-aT}{\sqrt{T}}\right) + e^{2ab}\Phi\left(\frac{b+aT}{\sqrt{T}}\right),$$

where $a = \sigma^{-1}(r - \frac{1}{2}\sigma^2)$. Then V_0 is equal to:

$$e^{-rT}\left((1+\alpha)F\Phi\left(\frac{\log\frac{F}{S_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - (1-\alpha)S_0\Phi\left(\frac{\log\frac{F}{S_0} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \alpha F\right)$$

where $\alpha = \sigma^2/2r$.

Double barriers

Some payoffs may involve all three prices — S_T, $S_*(T)$, and $S^*(T)$. An example of this is a double barrier call option struck at k which pays off only if the stock never goes below level c_1 and never goes above level c_2. That is

$$X = (S_T - k)^+ I(c_1 < S_*(T);\ S^*(T) < c_2).$$

The same principles apply as before. It is helpful to have a generalised reflection principle, which says that for W a $\mathbb{P}$-Brownian motion, with $W_0 = 0$,

$$\begin{aligned}\mathbb{P}\big(W_T \in dy;\ a < W_*(T);\ W^*(T) < b\big) \\ = \sum_{n\in\mathbb{Z}}\big\{p_T(2n(a-b), y) - p_T(2n(b-a), y-2a)\big\}\, dy.\end{aligned}$$

This can be converted into a $\mathbb{Q}$-probability simply by multiplying it by the change of measure factor $d\mathbb{Q}/d\mathbb{P} = \exp(ay - \frac{1}{2}a^2T)$, where $a = \sigma^{-1}(r - \frac{1}{2}\sigma^2)$.